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n-Dimensional Cross Product and Its Application to Matrix Eigenanalysis

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An extension to n-dimensional space of the cross-product concept by means of a definition derived from the Laplace expansion of the determinant is presented. Based on this definition, which evaluates a vector orthogonal to $(n-1)$ vectors in the n-dimensional space, some techniques for computing the matrix eigenvectors are developed. These algorithms allow one to compute an orthogonal eigenvector set for any eigenvalue algebraic multiplicity and for real or complex nondefective matrices. An algorithm for computing the generalized eigenvectors and the upper-diagonal coupling elements associated with an eigenvalue having a geometrical multiplicity equal to one (full-defective matrices) is also included. Numerical examples to clarify the proposed algorithms have been presented. Finally, the equivalent symmetric S-matrix, which has the same eigenvector set, and the skew-symmetric tilde matrix, which performs the cross product in n-dimensional space, are provided.

Introduction

SCIENTIFIC applications often require the use of special mathematical techniques not yet developed. Therefore, it often happens that scientists try to devise the proper mathematical technique for their applications. Sometimes these techniques can be generalized to a broader class of problems, of which the application is only a subset. This is exactly what occurred with Ref. 1. That paper, concerning optimal attitude estimation, develops a method arriving at the solution of Davenport's equation of the q-method algorithm.[2] This equation, which can be written as

$$\boldsymbol{K}\boldsymbol{q}_{\text{opt}} = \lambda_{\max}\boldsymbol{q}_{\text{opt}} \tag{1}$$

provides the optimal quaternion $\boldsymbol{q}_{\text{opt}}$ estimating spacecraft attitude as the eigenvector associated with the greatest eigenvalue $\lambda_{\max}$ of the known symmetric matrix $\boldsymbol{K}$. As demonstrated by QUEST,[3] the complete eigenanalysis of Eq. (1) is unnecessary. In fact QUEST evaluates $\lambda_{\max}$ by applying the Newton–Raphson iterative technique (which rapidly converges to the solution if $\lambda_0 = 1$ is chosen as the starting point) to the $\boldsymbol{K}$-matrix characteristic equation and then evaluates the associated eigenvector $\boldsymbol{q}_{\text{opt}}$ by an application of the Cayley–Hamilton theorem. In Ref. 1 the closed-form expression for $\lambda_{\max}$ is provided, and $\boldsymbol{q}_{\text{opt}}$ is computed in two different ways, one of which evaluates $\boldsymbol{q}_{\text{opt}}$ as being the direction perpendicular to all of the row vectors of the singular matrix $\boldsymbol{D} = \boldsymbol{K} - \lambda_{\max}\boldsymbol{I}$. In fact, Eq. (1) can also be written as $\boldsymbol{D}\boldsymbol{q}_{\text{opt}} = \boldsymbol{0}$ and, therefore, $\boldsymbol{q}_{\text{opt}}$ can be seen as perpendicular to the three-dimensional hyperplane where all of the $\boldsymbol{D}$-matrix row vectors lie. In doing so, Ref. 1 has introduced the formulation for computing the four-dimensional cross product among three vectors defined in a four-dimensional space. The idea of computing the eigenvectors in such a way is demonstrated here as applicable to any matrix, real and/or complex and/or defective.

Because the extension of the cross product to the n-dimensional space and its application to the eigenvectors computation of matrix eigenanalysis could be a subject of general interest and a useful tool for many technical fields, it cannot be restricted to the spacecraft attitude determination problem and, therefore, it is presented in its complete form by providing the general mathematical theory and by investigating interesting relationships with matrix eigenanalysis. The paper also shows that is possible to evaluate a symmetric matrix having the same set of eigenvectors of a nonsymmetric matrix and provides, without demonstration, the formula for the skew-symmetric tilde matrix, which performs the cross product in n-dimensional space.

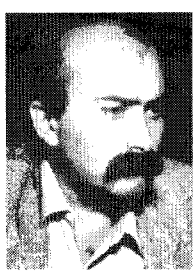
Note that, although the computation of a vector perpendicular to $(n-1)$ vectors in the n-dimensional space is an existing concept, which is defined in Grassman algebra[4,5] (the relevant presentation is given in the Appendix for completeness' sake), it is hereafter provided as an extension of the cross product concept and in a way that makes it accessible to those unfamiliar with Grassman algebra.

Definition

Let $\boldsymbol{D}$ be an $n \times n$ nonsingular matrix whose ith row vector is indicated by $\boldsymbol{d}_i^T$. The determinant of $\boldsymbol{D}$ can be computed by using the Laplace expansion by its generic ith row, that is, as a dot product between the vector $\boldsymbol{d}_i$ and a vector $\boldsymbol{v}_i$ as follows:

$$\begin{aligned}\det(\boldsymbol{D}) &= \sum_{k=1}^{n}(-1)^{i+k}\det(\boldsymbol{D}_{ik})\boldsymbol{d}_i(k) \\ &= \{\ldots,(-1)^{i+k}\det(\boldsymbol{D}_{ik}),\ldots\}\boldsymbol{d}_i = \boldsymbol{v}_i^T\boldsymbol{d}_i\end{aligned} \tag{2}$$

The $(n-1)\times(n-1)$ submatrices $\boldsymbol{D}_{ik}$ are obtained from $\boldsymbol{D}$ by deleting both the ith row $(\boldsymbol{d}_i^T)$ and the kth column. The $\boldsymbol{v}_i$ vector is completely



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independent of the d_i row vector because the submatrices D_{ik} are made up of the remaining d_k vectors (with $k \neq i$). From Eq. (2) it follows that the determinant of D becomes zero in three cases: 1) when $d_i = 0$, 2) when $v_i = 0$, and 3) when d_i is perpendicular to v_i. Case 1 is banal and does not add useful information; case 2, involving the vector v_i built with determinants of the $(n-1)\times(n-1)$ matrices D_{ik}, just moves the nth-order problem to order $(n-1)$ or lower. What is interesting for this study is case 3, that is, the case of d_i perpendicular to v_i. Therefore, let $\bar{D}$ be a simple $n \times n$ singular matrix; simple singular, that is, one-time singular, means that the matrix has rank $(n-1)$; this implies that it has only one zero singular value and that there is only one linear correlation among its row vectors. Matrix $\bar{D}$ could be obtained by substituting for d_i, in the D-matrix, the vector

$$\bar{d}_i = \sum_{k=1,\, k\neq i}^{n} \alpha_k d_k \tag{3}$$

Equation (3) introduces the linear correlation among the row vectors. In general, with the exceptions of $\bar{d}_i = 0$ (which, if all of the d_k are independent, could be obtained only if all of the α_k are zero) and of $v_i = 0$ (which implies a $\bar{D}$ matrix having rank lower than $n-1$), the resulting matrix $\bar{D}$ will be singular because $\bar{d}_i$ will be perpendicular to v_i:

$$\det(\bar{D}) = \bar{d}_i^T v_i = \sum_{k=1,\, k\neq i}^{n} \alpha_k d_k^T v_i = 0 \tag{4}$$

This expression is, however, always valid no matter what the coefficients α_k. This fact implies that the vector v_i must be perpendicular not only to the $\bar{d}_i$ vector but also to all of the d_k vectors and, therefore, to all of the row vectors of the matrix $\bar{D}$. Now, evaluating the determinant of $\bar{D}$ by using the expansion by another jth row (with $j \neq i$), the resulting associated v_j vector also will be perpendicular to all of the row vectors of the matrix $\bar{D}$. This implies that, for a simple singular matrix, all of the v_i vectors are perpendicular to all of the row vectors d_i^T and, as a consequence, all of the v_i vectors are parallel.

Let an $(n-1)\times n$ matrix be built with $(n-1)$ row vectors d_k^T and let $D^{[i]}$ be the $(n-1)\times(n-1)$ matrix obtained from it by deleting its ith column; therefore, because the expression

$$v = \left\{ \ldots, (-1)^{i+1} \det\left(D^{[i]}\right), \ldots \right\}^T \tag{5}$$

coincides in the three-dimensional space with the cross product, it can be considered as the mathematical representation of the cross product among the $(n-1)$ vectors d_k in the n-dimensional space. Hereafter, Eq. (5), which provides the n-dimensional cross-product v, will be denoted as the n-D$\times$-Π expression.

Application to the Computation of Matrix Eigenvectors

The given representation of n-D$\times$-Π can be used for computing the eigenvectors of a matrix (real or complex) with known eigenvalues (real or complex). In fact, the eigenproblem $Av = \lambda v$ can be written as $(A-\lambda I)v = Dv = 0$ and, therefore, the eigenvector v can be seen as the direction perpendicular to all of the row vectors of matrix D. As the problem $Dv = 0$ admits a nontrivial solution if and only if $\det(D) = 0$, the matrix D must be singular. If all of the n eigenvalues λ_i of A are different, then each $D_i = A-\lambda_i I$ matrix is one-time singular, which means that only one linear correlation condition, as that given in Eq. (3), is satisfied. Therefore, in this case each associated eigenvector v_i can be evaluated through the application of n-D$\times$-Π, as provided by Eq. (5), to the $(n-1)\times n$ matrix formed with $(n-1)$ row vectors of D, that is, excluding only one. The d_i^T row vector can be excluded from the set if and only if the matrix built with the $(n-1)$ remaining d_k^T row vectors $(k = 1, \ldots, n, k \neq i)$, has rank $(n-1)$. In this case not all of the n determinants of the $(n-1)\times(n-1)$ submatrices of Eq. (5) are zeros and, therefore, the trivial solution $v = 0$ is avoided.

The eigenvector matrix of A is then built, column by column, by applying the n-D$\times$-Π expression to all of the n different matrices D_i, that is, by computing, for each D_i, the corresponding v_i eigenvector. When the matrix A is real and has a complex eigenvalue $\lambda = \lambda_R + i\lambda_I$, the associated eigenvector $v = v_R + iv_I$ will be complex. Obviously, for this matrix, the conjugate eigenvector $v = v_R - iv_I$ is associated with the conjugate eigenvalue $\lambda = \lambda_R - i\lambda_I$.

The next section develops an algorithm that provides the solution when λ (real or complex) has an algebraic multiplicity $m > 1$, that is, when A has m eigenvalues λ and, therefore, m different eigenvectors are associated with λ.

Algebraic Multiplicity

Consider $AV = V\Lambda$ for a general nondefective $n \times n$ matrix A, where V and Λ are the eigenvector and the eigenvalue matrices, respectively. If m_i is defined as the algebraic multiplicity of λ_i, it follows that $\Sigma_i m_i = n$. In particular, the diagonal matrix Λ has the eigenvalues on its main diagonal $\mathrm{diag}(\Lambda) = \{\Lambda_1^T, \Lambda_2^T, \ldots, \Lambda_p^T\}^T$, where p is the number of different eigenvalues of A and Λ_i is an m_i-long vector of λ_i. The n-D$\times$-Π expression can be recursively used for computing each eigenvector matrix V_i as orthogonal. This implies the minimum value for its condition number, which is one. In fact, the condition number of a matrix is the ratio of the largest singular value to the smallest, and the orthogonal matrices have all of their singular values equal to one. For the purpose of inversion, for instance, a matrix is ill conditioned or poorly conditioned (small errors on data imply high errors on solution) if its condition number is large. A matrix with the condition number equal to one is called perfectly conditioned.

Hereafter, the n-D$\times$-Π application to the computation of m_i eigenvectors is restricted to the case of $m_i < n$, because if $m_i = n$ (all the λ_i are equal), then the A matrix becomes diagonal as $A = \lambda_i I$. In fact, because A is nondefective and the algebraic multiplicity of λ_i is n, then it follows that $\Lambda = \lambda_i I$. This implies $A = V\Lambda V^{-1} = \lambda_i I$, thereby demonstrating that $A = \Lambda = \lambda_i I$. For this matrix any vector is an eigenvector and, therefore, any matrix (e.g., the unity matrix), is an eigenvector matrix. Its computation is, therefore, unnecessary.

Now, if the eigenvalue λ of the matrix A has algebraic multiplicity $m < n$, then the matrix $D = A - \lambda I$ is m-times singular, which means that it has rank $(n-m)$. Let $M_0 = D(U_R, U_C)$ be one of such $(n-m)\times(n-m)$ nonsingular submatrices. The square matrix M_0 is formed with the d_{ij} elements identified by the $(n-m)$ rows (whose indices are stored in the used rows U_R integer vector) and by the $(n-m)$ columns (whose indices are stored in the used columns U_C integer vector) of matrix D. Matrix M_0 constitutes a basic matrix from which all of the eigenvectors set, associated with λ, can be built as explained hereafter. Let R_C be the m-long remaining columns integer vector of indices (those not included in U_C), that is, $U_C \cup R_C \equiv \{1, 2, \ldots, n-1, n\}$ and $U_C \cap R_C \equiv \{\emptyset\}$ (the empty set). The m unknown eigenvectors v_k are then computed starting with $k = 1$ by the following loop procedure, which will be referred to as the n-D$\times$-Π technique.

1) Choose one element $l \in R_C$.

2) Adjust the set sizes of U_C and R_C by moving the element l from R_C to U_C.

3) Build the $(n-m+k-1)\times(n-m+k)$ matrix D_k, which is obtained by inserting the elements defined by U_R rows of the l column into the matrix M_{k-1} at the proper column ascending-order place.

4) Solve, for the $(n-m+k)$ elements of vector t_k by using the n-D$\times$-Π expression provided by Eq. (5), the equation

$$D_k t_k = 0 \tag{6}$$

5) Normalize to one the t_k vector

$$\hat{t}_k = \frac{t_k}{\sqrt{t_k^T t_k}} \tag{7}$$

6) Set $v_k(U_C) = \hat{t}_k$, that is, place the $(n-m+k)$ elements of the unit-vector $\hat{t}_k$ into the v_k vector at the positions defined by the indices U_C, and set to zero all of the remaining elements, that is, $v_k(R_C) = 0$. The resulting n-elements vector v_k represents the kth normalized eigenvector of A.

7) Check if $k \geq m$; if so, the loop procedure is completed, otherwise continue.

8) Update matrix M_k by inserting, as a row, the $(n - m + k)$ elements of vector $v_k(U_C)$,

$$M_k = \left[D_k^T \quad v_k(U_C)\right]^T \tag{8}$$

9) Increase k and return to step 1.

The following numerical example is included to clarify the described procedure.

The matrix

$$D = \begin{bmatrix} 7 & -7.5 & -30 & -4.5 & -32.25 \\ -10 & 11 & 44 & 7 & 47 \\ -5 & 4.5 & 18 & 1.5 & 20.25 \\ 2 & -1 & -4 & 1 & -5.5 \\ 8 & -8 & -32 & -4 & -35 \end{bmatrix} \tag{9}$$

(where $D = A - \lambda_1 I$) is obtained from matrix A having the eigenvalues $\lambda_1 = 1$ with $m_1 = 3$ and $\lambda_2 = 2$ with $m_2 = 2$. It will be shown, just by way of example, how to compute all three eigenvectors associated with $\lambda_1 = 1$. For this case, $n = 5$ and $m = 3$. As is easily seen, any 2×3 submatrix of D has rank $(n - m) = 2$; therefore, let $M_0 = D(U_R, U_C)$ be the basic matrix, built with rows 3 and 5 ($U_R \equiv \{3, 5\}$ and $R_R \equiv \{1, 2, 4\}$) and columns 2 and 4 ($U_C \equiv \{2, 4\}$ and $R_C \equiv \{1, 3, 5\}$)

$$M_0 = \begin{bmatrix} 4.5 & 1.5 \\ -8 & -4 \end{bmatrix} \tag{10}$$

Starting with $k = 1$, element 1 is chosen from the remaining column indices set R_C. This element is then moved to the U_C set: that is, $U_C \equiv \{1, 2, 4\}$ and $R_C \equiv \{3, 5\}$. The system

$$D_1 t_1 = \begin{bmatrix} -5 & 4.5 & 1.5 \\ 8 & -8 & -4 \end{bmatrix} t_1 = 0 \tag{11}$$

leads to the solution $t_1 = \{-6, -8, 4\}^T$. The normalized vector is $\hat{t}_1 = \{-0.5571, -0.7428, 0.3714\}^T$ and the first eigenvector is $v_1 = \{-0.5571, -0.7428, 0, 0.3714, 0\}^T$. The next square matrix M_1 is

$$M_1 = \left[D_1^T \quad v_1(U_C)\right]^T = \begin{bmatrix} -5 & 4.5 & 1.5 \\ 8 & -8 & -4 \\ -0.5571 & -0.7428 & 0.3714 \end{bmatrix} \tag{12}$$

Increase k as $k = k + 1 = 2$ and choose element 3 from the remaining columns indices set R_C. This element is then moved to the U_C set: that is, $U_C \equiv \{1, 2, 3, 4\}$ and $R_C \equiv \{5\}$. The system

$$D_2 t_2 = \begin{bmatrix} -5 & 4.5 & 18 & 1.5 \\ 8 & -8 & -32 & -4 \\ -0.5571 & -0.7428 & 0 & 0.3714 \end{bmatrix} t_2 = 0 \tag{13}$$

leads to the solution $t_2 = \{17.8268, -19.3123, 10.7703, -11.8845\}^T$. The normalized vector is then $\hat{t}_2 = \{0.579, -0.6272, 0.3498, -0.386\}^T$, whereas the second is $v_2 = \{0.579, -0.6272, 0.3498, -0.386, 0\}^T$. The next square matrix M_2 is

$$M_2 = \left[D_2^T \quad v_2(U_C)\right]^T = \begin{bmatrix} -5 & 4.5 & 18 & 1.5 \\ 8 & -8 & -32 & -4 \\ -0.5571 & -0.7428 & 0 & 0.3714 \\ 0.579 & -0.6272 & 0.3498 & -0.386 \end{bmatrix} \tag{14}$$

Increase k as $k = k + 1 = 3$ and choose the last element from the remaining columns indices set R_C, which is 5. Element 5 is moved to the U_C set: that is, $U_C \equiv \{1, 2, 3, 4, 5\}$ and $R_C \equiv \{\emptyset\}$. The system

$$D_3 t_3 = \begin{bmatrix} -5 & 4.5 & 18 & 1.5 & 20.25 \\ 8 & -8 & -32 & -4 & -35 \\ -0.5571 & -0.7428 & 0 & 0.3714 & 0 \\ 0.579 & -0.6272 & 0.3498 & -0.386 & 0 \end{bmatrix} t_3 = 0 \tag{15}$$

leads to the solution $t_3 = \{-17.3435, 7.2427, 28.9708, -11.5299, -30.7896\}^T$. The normalized vector is $\hat{t}_3 = \{-0.3637, 0.1519, 0.6076, -0.2418, -0.6457\}^T$, which implies that the third (and last) eigenvector is $v_3 = \{-0.3637, 0.1519, 0.6076, -0.2418, -0.6457\}^T$. The resulting computed eigenvectors v_1, v_2, and v_3 are orthogonal; therefore, the eigenvector matrix $V = [v_1, v_2, v_3]$ has a condition number equal to one.

Complex Case

The n-D×-Π technique can be applied to a real or complex matrix as well as to a real matrix having complex conjugate eigenvalues. In these complex cases, however, software able to deal with the complex operation is needed. When such a software tool is not available, the complex case must be transformed into a form that considers only real operations, which is the subject of this section. The eigenanalysis of an $n \times n$ real matrix A_R having the complex conjugate eigenvalues $\lambda = \lambda_R \pm i\lambda_I$ with algebraic multiplicity $m(1 \leq m \leq \text{int}[n/2])$ can be written as

$$A_R(v_R + iv_I) = (\lambda_R + i\lambda_I)(v_R + iv_I) \tag{16}$$

which can be set in the real form

$$Dv = \begin{bmatrix} D_R & -D_I \\ D_I & D_R \end{bmatrix} \begin{Bmatrix} v_R \\ v_I \end{Bmatrix} = 0 \tag{17}$$

where D is an $(2n) \times (2n)$ matrix and where

$$D_R = A_R - \lambda_R I \qquad \text{and} \qquad D_I = -\lambda_I I \tag{18}$$

As can easily be observed, if $v = \{v_R^T, v_I^T\}^T$ is a solution of the system of Eq. (17), then $\bar{v} = \{v_I^T, -v_R^T\}^T$ will also be a solution of it. The $(2n) \times (2n)$ matrix D has rank $2(n - m)$, which ranges from n up to the value $2(n - 1)$. It is possible to evaluate the searched m orthogonal complex eigenvectors by applying the same n-D×-Π technique already described to the system $Dv = 0$, with few changes. Therefore, let $M_0 = D(U_R, U_C)$ be a $2(n - m) \times 2(n - m)$ nonsingular submatrix of D, formed with the elements identified by the U_R (used rows) and the U_C (used columns) indices vectors. The first $(k = 1)$ eigenvector $v_1 = \{v_{1R}^T, v_{1I}^T\}^T$ is then evaluated (and also the $\bar{v}_1 = \{v_{1I}^T, -v_{1R}^T\}^T$) by applying the n-D×-Π technique as described in points 1–6 in the preceding section. Then matrix M_1 is updated by adding the two solutions v_1 and $\bar{v}_1$ as follows:

$$M_1 = \left[D_1^T \quad v_1(U_C) \quad \bar{v}_1(U_C)\right]^T \tag{19}$$

The procedure is then iterated with the only exception that instead of adding one column (with index in R_C; see point 1) and computing one eigenvector at each loop, a couple of columns with indices in R_C must be added. Therefore, at each loop both matrices M_k and D_k increase in size by two; the eigenvector $v_k = \{v_{kR}^T, v_{kI}^T\}^T$ is actually computed, whereas the $\bar{v}_k = \{v_{kI}^T, -v_{kR}^T\}^T$ is simply derived from v_k. This procedure provides the m searched eigenvectors $v_{kR} + iv_{kI}$ associated with the eigenvalue $\lambda_R + i\lambda_I$ as orthogonal. The set of conjugate eigenvectors $v_{kR} - iv_{kI}$ is then associated with the conjugate eigenvalue $\lambda_R - i\lambda_I$, where $k = 1, \ldots, m$.

Alternatively, if $\det(D_I) \neq 0$, the system (17) can be written into the real system of order n

$$\begin{aligned} H_I v_R &= 0 \\ v_I &= R_I v_R \end{aligned} \qquad \text{or} \qquad \begin{aligned} H_I v_I &= 0 \\ v_R &= -R_I v_I \end{aligned} \tag{20}$$

where

$$R_I = D_I^{-1} D_R \qquad \text{and} \qquad H_I = R_I^2 + I \tag{21}$$

Matrix H_I has rank $(n - 2m)$ and, because $D_I = -\lambda_I I$, matrix D_I is nonsingular. The n-D×-Π technique can be applied to the system of Eq. (20) with only a few changes. Let $M_0 = H_I(U_R, U_C)$ be an $(n - 2m) \times (n - 2m)$ nonsingular submatrix of H_I, where U_R and U_C have the meaning given earlier. The real part of the first $(k = 1)$ eigenvector is then evaluated by applying the n-D×-Π technique from points 1–6 to the system $D_1 v_{1R} = 0$, where D_1 is obtained by inserting into the M_0 matrix the elements of H_I identified by the

U_R rows and the $l \in R_C$ column. The imaginary part is given by $v_{1I} = R_I v_{1R}$.

The procedure is then iterated with the exception that the matrix M_k is updated as

$$M_k = \begin{bmatrix} D_k^T & v_{kR}(U_C) & v_{kI}(U_C) \end{bmatrix}^T \tag{22}$$

at every loop, and a couple of columns l, $j(l, j \in R_C)$ update the D_k matrices. Therefore, both matrices M_k and D_k increase in size, at each loop, by two.

This alternate procedure has the undoubtable advantage, as compared with that using Eq. (17), of dealing with a matrix system of size n instead of $(2n)$; however, this alternate procedure does not provide an orthogonal set of eigenvectors. Numerical tests have shown, however, that the condition number of the eigenvector matrix so computed is quite good; further studies are needed to obtain the solution with the minimum condition number.

Similarly, if $\det(D_R) \neq 0$, the system (17) can be written in the real form of order n:

$$\begin{matrix} H_R v_R = 0 \\ v_I = R_R v_R \end{matrix} \quad \text{or} \quad \begin{matrix} H_R v_I = 0 \\ v_R = -R_R v_I \end{matrix} \tag{23}$$

where

$$R_R = D_R^{-1} D_I \qquad \text{and} \qquad H_R = R_R^2 + I \tag{24}$$

with analogous considerations as for the system of Eq. (20).

The next numerical example will clarify the described alternate procedure.

The matrix

$$A = \begin{bmatrix} -2/3 & 1/3 & -5/2 & -1/3 & 1/3 \\ 20/3 & -10/3 & 9 & -2/3 & -7/3 \\ 4 & -2 & 6 & 0 & -1 \\ -12 & 9 & -18 & 2 & 9/2 \\ 0 & 0 & -2 & 0 & 2 \end{bmatrix} \tag{25}$$

has the eigenvalue $\lambda_1 = 1 + i$ with $m_1 = 2$ (obviously $\lambda_2 = 1 - i$ with $m_2 = 2$) and $\lambda_3 = 2$ with $m_3 = 1$.

It will be shown how to compute the two eigenvectors associated with λ_1. In this case $n = 5$ and $m = 2$; therefore, it follows that $R_I = -D_I^{-1} D_R = I - A$ and

$$H_I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & -2 & 6 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & -6 & 0 & 2 \\ -8 & 4 & -12 & 0 & 4 \end{bmatrix} \tag{26}$$

which has rank $n - 2m = 1$. Starting with $k = 1$, let M_0 be the nonsingular submatrix of H_I with $U_R \equiv \{2\}$ and $U_C \equiv \{3\}$, that is, $M_0 = H_I(U_R, U_C) = [6]$. One element of $R_C \equiv \{1, 2, 4, 5\}$ is 1. Therefore, $U_C \equiv \{1, 3\}$ and $R_C \equiv \{2, 4, 5\}$; then we get the matrix $D_1 = [H_I(U_R, 1) \quad M_0] = [4 \ 6]$. System $D_1 t_1 = 0$ leads to the solution $t_1 = \{6, -4\}^T$ and, therefore, the real part of the first eigenvector is $v_{1R} = \{6, 0, -4, 0, 0\}^T$, whereas the imaginary part is $v_{1I} = R_I v_{1R} = \{0, -4, -4, 0, -8\}^T$. The normalized vectors $v_{1R} = \{0.4932, 0, -0.3288, 0, 0\}^T$ and $v_{1I} = \{0, -0.3288, -0.3288, 0, -0.6576\}^T$ represent the first complex eigenvector. The second $(k = k + 1 = 2)$ eigenvector is then evaluated by choosing a couple of elements (e.g., 2 and 4) from $R_C \equiv \{2, 4, 5\}$ set. This implies $U_C \equiv \{1, 2, 3, 4\}$ and $R_C \equiv \{5\}$ and $D_2^T = [H_I(U_R, U_C)^T \quad v_{1R}(U_C) \quad v_{1I}(U_C)]^T$ as

$$D_2 = \begin{bmatrix} 4 & -2 & 6 & 0 \\ 0.4932 & 0 & -0.3288 & 0 \\ 0 & 0.3288 & 0.3288 & 0 \end{bmatrix} \tag{27}$$

The system $D_2 t_2 = 0$ leads to $t_2 = \{0, 0, 0, -1.7297\}^T$ and, therefore, the real part of the second eigenvector is $v_{2R} = \{0, 0, 0, -1.7297, 0\}^T$ and the imaginary part is $v_{2I} = R_I v_{2R} = \{-0.5766, -1.1532, 0, 1.7297, 0\}^T$. The normalization leads to the vectors $v_{2R} = \{0, 0, 0, -0.6255, 0\}^T$ and $v_{2I} = \{-0.2085, -0.417, 0, 0.6255, 0\}^T$. At this point, because $k = k + 1 = 3 > m = 2$, the procedure ends.

Finally, for a complex matrix $A = A_R + iA_I$ with complex eigenvalue $\lambda = \lambda_R + i\lambda_I$ with algebraic multiplicity m, it is still possible to apply both techniques described earlier, provided that

$$D_R = A_R - \lambda_R I \qquad \text{and} \qquad D_I = A_I - \lambda_I I \tag{28}$$

Geometrical Multiplicity

Some matrices cannot be transformed into the diagonal form by means of a similarity transformation without achieving an associated singular eigenvector matrix. For these matrices, which are called defective, it is possible, however, to reach a quasidiagonal form, that is, the $N \times N$ Jordan form J, which can be partitioned with p different $n_i \times n_i$ Jordan blocks J_i $(i = 1, 2, \ldots, p; \Sigma_i n_i = N$, where n_i is the algebraic multiplicity of $\lambda_i)$. The J_i Jordan block contains the eigenvalue λ_i on its main diagonal $J_i(l, l) = \lambda_i$ $(l = 1, \ldots, n_i)$, while some coupling elements $J_i(k - 1, k) = s_k^{(i)}$ $(k = 2, \ldots, n_i$ and $s_1^{(i)} = 0)$ appear on the first upper diagonal, and zeros elsewhere.

Let J be the generic ith Jordan block. Associated with each $n \times n$ Jordan block J two kinds of eigenvectors are possible: the n_t true and the n_g generalized eigenvectors (where $n_t + n_g = n$). The n_t true eigenvectors are those associated with each $s_k = 0$. It follows that $1 \leq n_t < n$, because if $n_t = n$ then the J Jordan block is nondefective. The true eigenvectors satisfy the condition $(A - \lambda I)v_j = Dv_j = 0$, where $j = 1, \ldots, n_t$. Matrix D is, therefore, n_t times singular, and n_t also indicates the number of the singular values $\sigma_k = 0$ of D. If n_t, which is called the geometrical multiplicity of the eigenvalue λ, is one, the matrix is called full defective; otherwise, if $n_t > 1$, the matrix is called derogatory. The n_t true eigenvectors can be computed orthogonally by the n-D×-Π technique, already explained, applied to the system $Dv_j = 0$ (where $j = 1, \ldots, n_t$).

The n_g generalized eigenvectors are those associated with the coupling elements $s_k \neq 0$; each true eigenvector v_j (that is, each $s_k = 0$), has a set of $n^{(j)}$ generalized eigenvectors that could even be empty, which constitutes a chain of vectors satisfying the recursive chain relation

$$(A - \lambda I)w_k^{(j)} = Dw_k^{(j)} = s_k^{(j)} w_{k-1}^{(j)} \tag{29}$$

where $k = 1, \ldots, n^{(j)}$ and $j = 1, \ldots, n_t$, and, for $k = 1$, it results that $w_0^{(j)} = v_j$. The value of $n^{(j)}$ represents the length of the chain associated with the true eigenvector v_j. The chain relation associated with v_j implies

$$D^{k+1} w_k^{(j)} = 0 \tag{30}$$

where $k = 1, \ldots, n^{(j)}$. It will be shown how to evaluate, by a procedure based on n-D×-Π, the $(n - 1)$ generalized eigenvectors of an $n \times n$ full-defective matrix A associated with its only true eigenvector. A full-defective matrix A implies $s_k \neq 0$ $(k = 2, \ldots, n)$, $s_1 = 0$. Matrix $D = (A - \lambda I)$, derived from a full-defective matrix A, is full-defective too, with rank $(n - 1)$. Therefore, the n-D×-Π technique allows us to compute the only true eigenvector v_1 that satisfies the condition $Dv_1 = 0$.

Omitting the j index ($j = 1$ for full-defective matrices), the chain relation (29) can be written in the form

$$[D, \quad -w_{k-1}] \begin{Bmatrix} w_k \\ s_k \end{Bmatrix} = Q_k h_k = 0 \tag{31}$$

The $n \times (n + 1)$ matrix Q_k has rank $n - 1$, which is also the rank of D. Therefore, it is possible to apply the n-D×-Π technique to Eq. (31) as well as for computing a couple of solutions $h_k^{(1)}$ and $h_k^{(2)}$, which are orthogonal. In particular, because the $(n - 1) \times n$ matrix D_1 is made up of the elements identified by $(n - 1)$ vectors of D (where $R_R \equiv \{r\}$), the $h_k^{(1)}$ solution implies its last element as zero, that is, $s_k = 0$. Therefore, the associated computed eigenvector w_k, represented by the first n elements of the $h_k^{(1)}$ solution, must be equal to the true eigenvector v_1. Thus, because $h_k^{(1)} = \{v_1^T \ 0\}^T$, its computation is unnecessary.

Hence, Eq. (31) can be transformed into

$$\left[\boldsymbol{D}^{(r)}, -\boldsymbol{w}_{k-1}^{(r)}\right]\begin{Bmatrix}\boldsymbol{w}_k\\ s_k\end{Bmatrix} = \boldsymbol{Q}_k^{(r)}\boldsymbol{h}_k^{(2)} = \boldsymbol{0} \tag{32}$$

where the superscript (r) applied to $\boldsymbol{D}$ means that its r row has been replaced by the $\boldsymbol{v}_1^T$ vector, whereas when applied to the $\boldsymbol{w}_{k-1}$ vector it means that its r element has been set to zero. As can easily be observed, this substitution is such that it excludes from the system of Eq. (32) the solution associated with the true eigenvector $\boldsymbol{h}_k^{(1)} = \{\boldsymbol{v}_1^T \;\; 0\}^T$. This modification allows direct computation of the generalized eigenvector $\boldsymbol{w}_k$, as well as the upper-diagonal coupling elements s_k. If a unity s_k is needed, the computed generalized eigenvector $\boldsymbol{w}_k$ must then be modified according to $\boldsymbol{w}_k/s_k$.

The procedure can be clarified by the next numerical example. The 4×4 full-defective matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 & 1.5\\ 4 & 5 & 3 & 0.5\\ -8 & -8 & -4 & -2.5\\ -2 & -2 & -2 & 0\end{bmatrix} \tag{33}$$

has four eigenvalues $\lambda = 1$. Matrix $\boldsymbol{D} = \boldsymbol{A} - \lambda\boldsymbol{I}$ has rank $(n-1) = 3$. The only true eigenvector can be computed by applying the n-D$\times$-Π technique to the system $\boldsymbol{D}\boldsymbol{v}_1 = \boldsymbol{0}$. This can be done by selecting, for instance, the nonsingular matrix $\boldsymbol{M}_0 = \boldsymbol{D}(\boldsymbol{U}_R, \boldsymbol{U}_C)$, where $\boldsymbol{U}_R \equiv \{1, 2, 3\}$, $\boldsymbol{U}_C \equiv \{1, 2, 3\}$ and, therefore, $\boldsymbol{R}_R \equiv \{r\} \equiv \{4\}$. This leads to $\boldsymbol{v}_1 = \{0.7071, -0.7071, 0, 0\}^T$, which represents the true normalized eigenvector of matrix $\boldsymbol{A}$. The first generalized eigenvector $(k = 1)$ is computed by using Eq. (32).

The matrix (note that $\boldsymbol{w}_0 = \boldsymbol{v}_1$)

$$\left[\boldsymbol{D}^{(4)}, -\boldsymbol{w}_0^{(4)}\right] = \begin{bmatrix} 2 & 2 & 1 & 1.5 & -0.7071\\ 4 & 4 & 3 & 0.5 & 0.7071\\ -8 & -8 & -5 & -2.5 & 0\\ 0.7071 & -0.7071 & 0 & 0 & 0\end{bmatrix} \tag{34}$$

leads to the solution of $\boldsymbol{h}_1^{(2)} = \{0, 0, 1, -2, -2.8284\}^T$, which implies $\boldsymbol{w}_1 = \{0, 0, 1, -2\}^T$ and $s_1 = -2.8284$. To have a unity upper-diagonal element, the first generalized eigenvector is normalized according to $\boldsymbol{w}_1 = \boldsymbol{w}_1/s_1 = \{0, 0, -0.3536, 0.7071\}^T$.

The second generalized eigenvector $(k = 2)$ is then computed using the matrix

$$\left[\boldsymbol{D}^{(4)}, -\boldsymbol{w}_1^{(4)}\right] = \begin{bmatrix} 2 & 2 & 1 & 1.5 & 0\\ 4 & 4 & 3 & 0.5 & 0\\ -8 & -8 & -5 & -2.5 & 0.3536\\ 0.7071 & -0.7071 & 0 & 0 & 0\end{bmatrix} \tag{35}$$

which leads to the solution $\boldsymbol{h}_2^{(2)} = \{-1, -1, 2.5, 1, -2.8284\}^T$. This implies $\boldsymbol{w}_2 = \{-1, -1, 2.5, 1\}^T$ and $s_2 = -2.8284$. Here $\boldsymbol{w}_2$ is then normalized according to $\boldsymbol{w}_2 = \boldsymbol{w}_2/s_2 = \{0.3536, 0.3536, -0.8839, -0.3536\}^T$.

Finally, the third generalized eigenvector $(k = 3)$ is computed using the matrix

$$\left[\boldsymbol{D}^{(4)}, -\boldsymbol{w}_2^{(4)}\right] = \begin{bmatrix} 2 & 2 & 1 & 1.5 & -0.3536\\ 4 & 4 & 3 & 0.5 & -0.3536\\ -8 & -8 & -5 & -2.5 & 0.8839\\ 0.7071 & -0.7071 & 0 & 0 & 0\end{bmatrix} \tag{36}$$

which implies the solution $\boldsymbol{h}_3^{(2)} = \{0, 0, -0.25, -0.5, -2.8284\}^T$, that is, $\boldsymbol{w}_3 = \{0, 0, -0.25, -0.5\}^T$ and $s_3 = -2.8284$. The third generalized eigenvector is then normalized as $\boldsymbol{w}_3 = \boldsymbol{w}_3/s_3 = \{0, 0, 0.0884, 0.1768\}^T$.

The condition of having unity upper-diagonal elements leads to an alternate procedure, based on the fact that the $\boldsymbol{D}^{(r)}$ matrix has full rank. Equation (32) can be set up to the recursive equation

$$\boldsymbol{w}_k = \left[\boldsymbol{D}^{(r)}\right]^{-1}\boldsymbol{w}_{k-1}^{(r)} \tag{37}$$

simply derived from Eq. (32) by setting $s_k = 1$ (where $k = 2, \ldots, n$).

This paper does not include a solution technique for the derogatory matrices (which would complete the analysis for defective matrices), because it is still unknown.

Equivalent Symmetric Matrix

The n-D$\times$-Π technique has been applied to the computation of matrix eigenvectors. This has been done by modifying the original problem to the equivalent one:

$$\boldsymbol{D}\boldsymbol{v} = \boldsymbol{0} \tag{38}$$

where the $n \times n$ matrix $\boldsymbol{D}$ has rank $(n - m)$ and m is the number of eigenvectors to be found, that is, those associated with the m zero eigenvalues of $\boldsymbol{D}$. In general, matrix $\boldsymbol{D}$ is nonsymmetric. In this section it is shown that it is possible to build an $n \times n$ symmetric matrix $\boldsymbol{S}$, which has the same set of m eigenvectors associated with m zero eigenvalues as $\boldsymbol{D}$. Therefore, $\boldsymbol{D}$ can be replaced by $\boldsymbol{S}$ when the preferred form is symmetric. Note that a real matrix $\boldsymbol{A}$ having complex conjugate eigenvalues has a complex associated $\boldsymbol{D}$ matrix; then, the resulting equivalent symmetric matrix $\boldsymbol{S}$ will be complex as well.

Let $\boldsymbol{U}_R$ be the indices set of the $(n - m)$ row vectors of $\boldsymbol{D}$ constituting an $(n - m) \times n$ matrix with rank $(n - m)$. Then it is possible to write

$$\sigma_k = \boldsymbol{v}^T\boldsymbol{d}_k = \boldsymbol{d}_k^T\boldsymbol{v} = 0 \qquad (\forall k \in \boldsymbol{U}_R) \tag{39}$$

Therefore, the quadratic quantity

$$\sigma^2 = \sum_{\forall k \in \boldsymbol{U}_R}\left(\boldsymbol{v}^T\boldsymbol{d}_k\right)\left(\boldsymbol{d}_k^T\boldsymbol{v}\right) = \boldsymbol{v}^T\boldsymbol{S}\boldsymbol{v} \tag{40}$$

is zero at the m solutions and positive elsewhere, where the symmetric matrix

$$\boldsymbol{S} = \sum_{\forall k \in \boldsymbol{U}_R}\boldsymbol{d}_k\boldsymbol{d}_k^T \tag{41}$$

has been introduced.

An $n \times n$ matrix built as $\boldsymbol{S}$ is m *times singular* if all of the $(n - m)$ vectors $\boldsymbol{d}_k$ constitute an $(n - m)$ independent vectors basis, which is our case. In particular, such a matrix has nonnegative eigenvalues coincident with its singular values whereas the eigenvectors matrix is equal to both its right and left eigenvector matrices.

The quantity σ^2 is a sum of nonnegative terms, which becomes zero (minimum value) at any direction of $\boldsymbol{v}$ perpendicular to all of the $\boldsymbol{d}_k$ row vectors. Therefore, $\boldsymbol{v}$ can be evaluated as the unit vector satisfying $\min \sigma^2 = 0$. In this minimization the constraint for $\boldsymbol{v}$ to be a unit vector $(\boldsymbol{v}^T\boldsymbol{v} = 1)$ must be included. This leads to the condition of minimizing the augmented cost function, which can be written as

$$\min_{\boldsymbol{v}} \sigma_*^2 = \boldsymbol{v}^T\boldsymbol{S}\boldsymbol{v} - \xi(\boldsymbol{v}^T\boldsymbol{v} - 1) = 0 \tag{42}$$

where ξ is the Lagrangian multiplier.

The solution of Eq. (42) is obtained by setting

$$\frac{\mathrm{d}\sigma_*^2}{\mathrm{d}\boldsymbol{v}} = 2(\boldsymbol{S}\boldsymbol{v} - \xi\boldsymbol{v}) = \boldsymbol{0} \tag{43}$$

which states that the vector $\boldsymbol{v}$ is the eigenvector of the matrix $\boldsymbol{S}$ associated with the eigenvalue ξ. This eigenvalue can be computed by premultiplying Eq. (43) by $\boldsymbol{v}^T$:

$$\boldsymbol{v}^T\boldsymbol{S}\boldsymbol{v} = \xi = \sigma^2 = 0 \tag{44}$$

Consequently, the unit vector $\boldsymbol{v}$, which is an eigenvector of $\boldsymbol{D}$ associated with its zero eigenvalue, is also an eigenvector of the symmetric

matrix S associated with its zero eigenvalue. Therefore, D can be replaced by S (no matter what the algebraic multiplicity of ξ) when the preferred form is symmetric.

n-D×-Π Skew-Symmetric Matrix

The three-dimensional cross product $u \times v$ can be expressed by the skew-symmetric tilde matrix $\tilde{u}$:

$$u \times v = \tilde{u}v = \begin{bmatrix} 0 & -u(3) & u(2) \\ u(3) & 0 & -u(1) \\ -u(2) & u(1) & 0 \end{bmatrix} \begin{Bmatrix} v(1) \\ v(2) \\ v(3) \end{Bmatrix} \tag{45}$$

It is possible to extend the skew-symmetric 3×3 tilde matrix $\tilde{u}$ to n-dimensional space. The resulting $n \times n$ skew-symmetric matrix $\tilde{U}$, which performs n-D×-Π:

$$v_1 \times v_2 \times \cdots \times v_{n-2} \times v_{n-1} = \tilde{U}v_{n-1} \tag{46}$$

involves $(n-2)$ n-element vectors.

Without a demonstration, the generalized skew-symmetric tilde matrix $\tilde{U}$ is provided by the formula

$$\tilde{U}(i, j) = (-1)^{n-i-j} \det(V_{ij}) = -\tilde{U}(j, i) \qquad 1 \le i < j \le n$$
$$\tilde{U}(i, i) = 0 \qquad 1 \le i \le n \tag{47}$$

where the $(n-2) \times (n-2)$ submatrices V_{ij} are obtained by deleting the ith and the jth column vectors from the $(n-2) \times n$ matrix V, where

$$V^T = [v_1, v_2, \ldots, v_{n-2}] \tag{48}$$

Conclusions

An ad hoc expression for the vector cross product in an n-dimensional space is presented, which has been identified as n-D×-Π and derived from the Laplace expansion of the determinant of an $n \times n$ matrix. The n-D×-Π expression is then used for the computation of the eigenvectors for a nondefective matrix. The resulting algorithm, called the n-D×-Π technique, allows the computation of a set of orthogonal eigenvectors associated with an eigenvalue with any algebraic multiplicity. This technique can be applied to a real or complex matrix as well as to a real matrix having complex conjugate eigenvalues. When a software tool able to deal with complex operation is not available, two different algorithms are provided for the complex problem, which considers only real operations. One of them provides the computation of orthogonal eigenvectors. Then, to extend the application to defective matrices, an algorithm for computing all of the generalized eigenvectors associated with a true eigenvector of a full-defective matrix, as well as the upper-diagonal coupling elements, has been developed. An n-D×-Π-based solution technique (which would complete the study for defective matrices) for the derogatory matrices is not included, since it has not been found. Finally, the equivalent symmetric matrix, which has the same eigenvectors set associated with the zero eigenvalue of a nonsymmetric matrix, has been introduced and, without including a demonstration, the mathematical expression of the skew-symmetric $n \times n$ tilde matrix performing n-D×-Π has been given. Because of the complexity of the proposed techniques and for the sake of clarity, numerical examples are included to emphasize the devised procedures. Comparisons with respect to the relevant existing techniques are not included, leaving this task to a subsequent paper, which will define the advantages and disadvantages of the proposed algorithms. At present, the proposed techniques represent an interesting alternate approach, which might be considered a useful tool in many technical fields, as, for instance, the application (optimal attitude estimation) that originated this study. However, the n-D×-Π application is not restricted to the computation of matrix eigenvectors but can be used for giving a solution to all of the mathematical problems that can be reduced to the linear form $Dv = 0$, where D is any $m \times n$ matrix ($m \le n$) with rank $r \le m$ and v is the n-element unknown vector(s) to be computed. In other words, the n-D×-Π expression and technique can be used to solve the mathematical problems whose vectorial solutions are perpendicular to some given vectors, in any n-dimensional space.

Appendix: Vector Cross Product as Defined in Grassman Algebra

The computation of the direction perpendicular to $(n-1)$ vectors in the n-dimensional space is an existing mathematical object, which is known, at least to differential geometers. The n-D×-Π expression provided and derived by the Laplace expansion of the determinant is only an ad hoc expression, which can be directly understood even by readers who do not have any knowledge of exterior algebra and differential forms. Even though it is possible to show that exterior algebra is just the Laplace expansion of the determinant, the associated formalism is a little unusual for many readers and sometimes difficult. Therefore, the n-D×-Π expression can be considered only a useful subset of the general definition (see paragraph 6.2 of Ref. 4 and Ref. 5), which involves Grassman algebra with its operators. For completeness, this is given in the following.

Let $r_1, r_2, \ldots, r_{n-1}$ be $(n-1)$ vectors in R_n (n-dimensional real space). Let $\hat{}$ denote the wedge product (see Ref. 4, paragraph 6.1) of Grassman algebra of R_n and * the Hodge star operator (Ref. 4, paragraph 6.2) on the Grassman algebra of R_n with the standard metric given by the dot product. Then, the cross product of $r_1, r_2, \ldots, r_{n-1}$ is given by

$$^*(r_1 \hat{} r_2 \hat{} \cdots \hat{} r_{n-2} \hat{} r_{n-1}) \tag{A1}$$

In fact, this can be generalized to any k vectors, where $k < n$, although the resulting object is an $(n-k)$ form and not a vector. Obviously, when $k = n-1$ it is a $n-(n-1) = 1$ form, which is a vector. Next proof of how the $^*(r_1 \hat{} r_2 \hat{} \cdots \hat{} r_{n-1})$ is orthogonal to r_i for $i = 1, \ldots, n-1$, is given. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of R_n. Let $L_{(k)}$ denote the space of k vectors on R_n in the exterior algebra. That is, elements of $L_{(k)}$ are made up of linear combinations of objects of the following form:

$$r_1 \hat{} r_2 \hat{} \cdots \hat{} r_{k-1} \hat{} r_k \tag{A2}$$

where r_i is a vector in R_n. When $k = 0$, $L_{(0)}$ is just R.

Then $^*: L_{(k)} \to L_{(n-k)}$ as follows:

$$\begin{aligned} ^*(1) &= \operatorname{sgn} e_1 \hat{} e_2 \hat{} \cdots \hat{} e_n && \text{in } L_{(n)} \\ ^*(e_1 \hat{} e_2 \hat{} \cdots \hat{} e_n) &= \operatorname{sgn} 1 && \text{in } R \\ ^*(e_1 \hat{} e_2 \hat{} \cdots \hat{} e_k) &= \operatorname{sgn} e_{k+1} \hat{} e_{k+2} \hat{} \ldots \hat{} e_n && \text{in } L_{(n-k)} \end{aligned} \tag{A3}$$

The sgn is the sign of the orientation of $e_1 \hat{} e_2 \hat{} \cdots \hat{} e_n$, where sgn $= +1$ if the orientation is positive and sgn $= -1$ if the orientation is negative.

Let $w = ^*(r_1 \hat{} r_2 \hat{} \cdots \hat{} r_{n-1})$, where r_i $(i = 1, \ldots, n-1)$ are vectors in R_n. Because $^*: L_{(k-1)} \to L_{(1)} = R_n$, w is then a vector in R_n. Let $\langle , \rangle$ denote the inner product in R_n. This can be extended to $L_{(k)}$ in general (see Ref. 4, paragraph 6.2.11). Assuming this, let r be any one of the vectors $\{r_1, r_2, \ldots, r_{(n-1)}\}$. Now look at $\langle r, w \rangle$ and see if this is 0 as we expect inasmuch as w is the cross product of the r_i and r is one of the r_i. There are two identities with regard to the Hodge star and wedge products and inner products:

$$\langle r, w \rangle = ^*(r \hat{} ^* w) \tag{A4}$$

and

$$^{**} = (-1)^{k(n-k)} \qquad \text{on} \qquad L_{(k)} \tag{A5}$$

In our case, because $k = n-1$, we have

$$^{**} = (-1)^{(n-1)} \qquad \text{on} \qquad L_{(n-1)} \tag{A6}$$

This means

$$^*w = (-1)^{(n-1)} r_1 \hat{} r_2 \hat{} \cdots \hat{} r_{n-1} \tag{A7}$$

thus,

$$\langle \boldsymbol{r}, \boldsymbol{w} \rangle = (\text{sign}) \langle \boldsymbol{r}, \boldsymbol{r}_1 \hat{} \boldsymbol{r}_2 \hat{} \cdots \hat{} \boldsymbol{r}_{n-1} \rangle = 0 \qquad \text{(A8)}$$

because $\boldsymbol{r}$ is one of the $\boldsymbol{r}_i$.

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